M1220-2 **Final Examination** **Spring 2005**

Final Examination Scores (out of 136): mean = 64

25th percentile = 53; median (50th percentile) = 66; 75th percentile = 79

Final Grades:

The final grade in the course = .15(QuizAverage) + .15(WebWorksTotal) + .40(MidtermAverage) + .30(FinalExaminationScore).

Final Grade Scores (out of 136): mean = 74;

25th percentile = 65; median (50th percentile) = 76; 75th percentile = 84

The minimum score for each letter grade was: A: (91) A-: (87) B+: (83) B: (77) B-: (73) C+: (67) C: (63) C-: (57) D+: (53) D: (48) D-: (40)

E: below 40

I will be in my office Monday afternoon, May 16th, if you want to pick up your final now. Otherwise, you can pick it up any time during Fall Semester 2005. If you have a question concerning your final grade you can email me at aroberts@math.utah.edu.

1. (15 pts) Find $\frac{dy}{dx}$ in each of the following. You do not need to simplify your answer.

i)

$$y = \arctan(e^{4x})$$

$\frac{dy}{dx} = \frac{4e^{4x}}{1+(e^{4x})^2} = \frac{4e^{4x}}{1+(e^{8x})}$

ii)

$$y = \ln[\sin(x^2+1)]$$

$\frac{dy}{dx} = \frac{2x\cos(x^2+1)}{\sin(x^2+1)}$

iii)

$$y = [\cos(2x)]^{x^2}$$

$\ln y = x^2 \ln[\cos(2x)]$

$\frac{1}{y}\frac{dy}{dx} = x^2\left[\frac{-2\sin(2x)}{\cos(2x)}\right] + 2x\ln[\cos(2x)].$

$\frac{dy}{dx} = [\cos(2x)]^{x^2}\left[\frac{-2x^2\sin(2x)}{\cos(2x)} + 2x\ln[\cos(2x)]\right].$

2. (20 pts) Evaluate the following integrals. Give a numerical answer for the definite integrals, evaluating the trignometric functions of common angles.

i.

$$\int \frac{\sin(\sqrt{x})}{\sqrt{x}} dx$$

Let $u = \sqrt{x}$, then $du = \frac{dx}{2\sqrt{x}}$ and the integral becomes:

$2\int \frac{\sin 2\sqrt{x}}{2\sqrt{x}} dx = 2\int \sin u du = -2\cos\sqrt{x} + C.$

ii.

$$\int \ln(2x^3)dx$$

Use integration by parts and let $u = \ln(2x^3)$, and $dv = dx$. Then, $du = \frac{3}{x}dx, v = x$. So the above integral equals,
$x\ln(2x^3) - \int 3dx = x\ln(2x^3) - 3x + C$.

iii.

$$\int_3^5 \frac{1}{x^2 - 6x + 13}dx$$

$\int_3^5 \frac{1}{x^2-6x+13}dx = \int_3^5 \frac{dx}{(x-3)^2+4} = \int_3^5 \frac{dx}{4\left[(\frac{x-3}{2})^2+1\right]}$.
Let $u = \frac{x-3}{2}, du = \frac{dx}{2}$ and the above integral becomes
$\frac{1}{2}\int_0^1 \frac{du}{u^2+1} = \frac{1}{2}\text{arctan}u|_0^1 = \pi/8$.

iv)

$$\int_0^1 \frac{\sqrt{x}}{x+1}dx$$

Let $u = \sqrt{x}, u^2 = x, 2udu = dx$ so that $\int_0^1 \frac{\sqrt{x}}{x+1}dx = 2\int_0^1 \frac{u^2}{u^2+1}du$. Dividing $u^2 + 1$ into u^2 this integral becomes
$2\int_0^1 1 - \frac{1}{u^2+1}du = 2[1 - \pi/4]$.

3. (12 pts) Determine whether the following series **diverge, converge conditionally, or converge absolutely**. Justify your answer completely.

i.

$$\sum_{1}^{+\infty}(-1)^n \frac{n}{e^{3n^2}}$$

In the integral, $\int_1^{+\infty} xe^{-3x^2}dx$, if we let $u = 3x^2, du = 6xdx$, then the integral becomes $\frac{1}{6}\int_3^{+\infty} e^{-u}du = \frac{1}{6}e^{-3}$. Since the integral converges, by the Integral Test, the above series converges absolutely.

ii.

$$\sum_{1}^{+\infty}(-1)^n \frac{n+1}{\ln n}$$

Using L'Hopital's Rule we see that $\lim_{x\to+\infty} \frac{x+1}{\ln x} = +\infty$. This means that the terms in the sequence $\{a_n\}$ where $a_n = (-1)^n \frac{n+1}{\ln n}$ do not approach zero. By the Nth term Divergence Test, the series $\sum a_n$ cannot converge.

4. (15 pts) **Power Series:** Find the convergence set for the following two power series.

i. Find the convergence set for the following power series

$$\sum_{1}^{+\infty} \frac{(x-2)^n}{n2^n}$$

Note that $\lim_{n\to+\infty} \left| \frac{(x-2)^{n+1}}{(n+1)2^{n+1}} \frac{n2^n}{(x-2)^n} \right|$

$= \left|\frac{x-2}{2}\right| \lim_{n\to+\infty} \frac{n}{n+1} = \frac{|x-2|}{2}$. So we want $|x-2| < 2$ or $0 < x < 4$. When $x = 0$, the series becomes the alternating harmonic series which converges. When $x = 4$, the series becomes the harmonic series which diverges. So the convergence set for this series is $[0, 4)$.

ii. Find the Taylor polynomial of order 3 based at 1 for the function $f(x) = e^{2x}$.

Note that $f^{(n)}(1) = 2^n e^2$ for all n so the Taylor Polynomial of order 3 is:
$g(x) = e^2 + \frac{2e^2(x-1)}{1!} + \frac{2^2e^2(x-1)^2}{2!} + \frac{2^3e^2(x-1)^3}{3!}$.

iii. Using the Maclaurin series for the function $f(x) = \frac{1}{1-x}$, find the Maclaurin series for the function $g(x) = \frac{x}{(1-x)^2}$. Either describe the series using summation notation or describe your series in the format: $b_1 + b_2 + b_3 + b_4 + b_5 + \ldots$. On what interval will this series converge to the function g?

$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots, |x| < 1;$
$\frac{d\left[\frac{1}{1-x}\right]}{dx} = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \ldots, |x| < 1;$
$x\frac{\frac{1}{1-x}}{dx} = \frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \ldots, |x| < 1.$

5. (16 pts)Limits and Indeterminate forms:

i. Find the following limits if the limit exists. If the limit doesn't exist, explain briefly why.

$$\lim_{n\to+\infty}\left(\frac{-2}{\pi}\right)^n$$

This sequence is of the form $\{r^n\}$ where $|r| < 1$ so the limit of the sequence is zero.

$$\lim_{x\to 0^+}(\cos x + x)^{\frac{1}{2x}}$$

$\lim_{x\to 0^+}(\cos x + x)^{\frac{1}{2x}} = \lim_{x\to 0^+}\frac{\ln(\cos x + x)}{2x}$
$= \lim_{x\to 0^+}\frac{-\sin x + 1)}{2(\cos x + x)} = 1/2$. So, $\lim_{x\to 0^+}(\cos x + x)^{\frac{1}{2x}} = e^{1/2}$

ii. Evaluate the following improper integrals. If the integral diverges explain briefly why. If the integral converges give it's numerical value.

$$\int_0^{+\infty} \frac{e^x}{1+e^{2x}} dx$$

Letting $u = e^x, du = e^x dx$ the above integral becomes $\lim_{b\to+\infty} \int_1^b \frac{1}{1+u^2} du = \lim_{b\to+\infty}[\arctan u]|_1^b] = \pi/2 - \pi/4 = \pi/4$.

$$\int_2^{10} \frac{1}{(x-2)^{2/3}} dx$$

$\int_2^{10} \frac{1}{(x-2)^{2/3}} dx = \lim_{b\to 2^+} \int_b^{10} \frac{1}{(x-2)^{2/3}} dx$
$= \lim_{b\to 2^+} \left[3(x-2)^{1/3}|_b^{10}\right] = \lim_{b\to 2^+} \left[6-(b-2)^{1/3}\right] = 6.$

6. (6 pts) Do **ONE** of the following problems. If you attempt more than one, indicate clearly which is to be counted for credit.

A: A bacteria population is growing at a rate that is proportional to its size. Initially, it is 2,000 and after 6 days it is 5,500. How many days will it take for the population to grow from 5,500 to 15,000? Leave your answer in terms of the natural logarithm.

From the given information, $5500 = 2000e^{6k}$, and solving for k we have, $k = \frac{\ln(55/20)}{6}$. Then solving for t in the equation, $15000 = 5500e^{kt}$, we have the time t needed to reach 15000 from 5500 is $t = \frac{\ln(150/55)}{k} = \frac{6\ln(150/55)}{\ln(55/20)}$.

B: Suppose a ball is dropped from a height of 10 feet and that each time it hits the floor, it rebounds to 1/3 of its previous height. Find the total distance the ball travels before it comes to rest.

The total distance travelled by the ball is $10 + 2\frac{1}{3}10 + 2(\frac{1}{3})^2 10 + 2(\frac{1}{3})^3 10 + \ldots$.
The sum equals $10 + 20(\frac{1}{3} + (\frac{1}{3})^2 + (\frac{1}{3})^3 + \ldots)$
$= 10 + 20[\frac{\frac{1}{3}}{1-\frac{1}{3}}] = 20$ feet.

C: Use the Trapezoid Rule to approximate the area
of the field shown to the right. All measurements are in feet.

Here $h = 3$ so the area of the field is $3/2[8 + 2(12 + 14 + 6) + 5] = 231/2\text{ft}^2$.

7. (6 pts) Do **ONE** of the following problems. If you attempt more than one, indicate clearly which is to be counted for credit.

A: Suppose that $g(x) = \int_0^x 3t^2 + 2dx$. Find: $g(1); g'(x); (g^{-1})'(3)$.

$g(1) = \int_0^1 3t^2 + 2dx = 3$.

By the Fundamental Theorem of Calculus, $g'(x) = 3x^2 + 2$.

From the Inverse Function Theorem we have, $(g^{-1})'(3) = \frac{1}{g'(g^{-1}(3))} = \frac{1}{g'(1)} = 1/5$.

B: Draw the graph of the curve in polar coordinates given by $r = 5\sin\theta$ and then find the area of the region enclosed by this curve.

The graph is a circle of radius (5/2) with center at (0, 5/2) in the rectangular coordinate system. The enclosed region has area $= (5/2)^2\pi = (25/4)\pi$.

8. (10 pts) Determine whether the following statements are True or False. If a statement is False, give a counterexample that shows why the statement is False.

i. If an infinite series diverges, then the terms in the series do not approach zero.

False: Consider the series, $\sum \frac{1}{n}$. This series diverges yet the terms in this series do approach zero.

ii. If $\int_1^{+\infty} \frac{1}{x^p} dx$ diverges, then $\int_0^1 \frac{1}{x^p} dx$ must converge.

False: $\int_1^{+\infty} \frac{1}{x} dx$ and $\int_0^1 \frac{1}{x} dx$ both diverge.

iii. In the Polar Coordinate system each point in the plane is represented by a unique ordered pair of real numbers.

False: $(\sqrt{2}, \pi/4), (-\sqrt{2}, 5\pi/4), (\sqrt{2}, 9\pi/4)$ all represent in polar coordinates the point $(1, 1)$ described in rectangular coordinates.

iv. If f is an even function and $\int_0^{+\infty} f(x)dx$ converges, then $\int_{-\infty}^{+\infty} f(x)dx$ converges.

True

v. If If the sequence $\{|a_n|\}$ has a limit, then the sequence $\{a_n\}$ also has a limit.

False: Consider the sequence $\{(-1)^n\}$. The sequence of absolute values, 1, 1 , 1, 1, 1 . . . has the limit 1 but the sequence itself, -1, 1, -1, 1, -1, . . . does not have a limit.